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Spinning Particles in Curved Spacetime

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We briefly discuss the relevant equations for the motion of spinning particles in curved spacetime. We describe the generalized Killing equations for spinning spaces and derive the constants of motion. We apply the formalism to solve for the motion of a pseudoclassical spinning particle in Schwarzschild–de Sitter spacetime.

KEY WORDS: pseudoclassical model; Killing–Yano tensors; supersymmetries; geodesics.

1. INTRODUCTION

The models of relativistic particles with spin have been proposed for a long time. The pioneer work concerning the Lagrangian description of the relativistic particle with spin was done by Frenkel in 1926 (Frenkel, 1926) and after that the literature on it grew vast (Frydryszak, 1996).

The relativistic spin one half particle models involving anticommuting vectorial degrees of freedom are usually called the spinning particles.

The action of spin one half relativistic particle with spinning degrees of freedom characterized by Grassmann (odd) variables was first proposed by Berezin and Marinov (1975, 1977) and soon after that was discussed and investigated by many authors (Balachandran *et al.*, 1977; Barducci *et al.*, 1976; Brink *et al.*, 1976, 1977; Casalbuoni, 1976).

In spite of the fact that the anticommuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians for these models have a natural interpretation in the context of the path-integral description of the quantum dynamics. The pseudoclassical equations acquire physical meaning when averaged over the inside of the functional integral (Barducci *et al.*, 1981b; Berezin and Marinov, 1975, 1977). In this semiclassical regime, neglecting higher order quantum correlations, it should be admissible to replace appropriate combinations of Grassmann spin–variables by real numbers. Using these ideas the motion of spinning particles in external fields have been investigated in Barducci *et al.* (1977, 1981a),

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Berezin and Marinov (1975, 1977), and van Holten (1991, 1992). On the other hand, generalizations of Riemannian geometry based on anticommuting variables have been proved to be of wide mathematical interest; for example, supersymmetric point particle mechanics has found applications in the area of index theorem, while the BRST methods are widely used in the study of topological invariants. Therefore, the study of motion of spinning particles in curved spacetime is well motivated.

In this paper we present an analysis of the motion of a pseudoclassical spinning particle in curved spacetime. We investigate the generalized Killing equations for the configuration space of spinning particles (*spinning space*) and describe the constants of motion along with a new kind of supersymmetries generated by the mysterious Killing–Yano-type square root of the Stackel–Killing tensor. Spacetime supersymmetry has previously been applied to charged black holes in the context of $N = 2$ supergravity theory. The new kind of supersymmetries addressed in this paper seems at first sight to be unrelated to that work. Actually, the new (*nongeneric*) supersymmetry related to the motion of spinning particles are applicable to all members of black hole spacetimes, while the Killing spinors giving rise to symmetries of the solutions of supergravity field equations arise only in the case of extreme solutions (or indeed naked singularities) whose mass and charge in suitable units are equal.

We apply the formalism to solve for the motion of a pseudoclassical spinning particle in a asymptotically de Sitter spacetime described by the Schwarzschild–de Sitter metric. This spacetime is interesting in that it contains a cosmological constant. In recent years there has been a renewed interest in cosmological constant as it is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a stage where it is geometrically similar to de Sitter spacetime (Guth, 1981). Among other things inflation has led to the cold dark matter. According to cold dark matter theory, the bulk of the dark matter is in the form of slowly moving particles (axions or neutralinos). If the cold dark matter theory proves correct, it would shed light on the unification of forces (Turner, 1995, 1998). In view of these interests in the cosmological constant this work is interesting.

The plan of this paper is as follows. In Subsection 2.1 we summarize the relevant equations for the motion of spinning particles in curved spacetime and briefly discuss their physical interpretation. In Subsection 2.2 we review Noether's theorem and the generalized Killing equations for spinning spaces. In Subsection 2.3 we describe the derivation of the constants of motion, which exist in any theory, in terms of the solutions of the generalized Killing equations. In Subsection 2.4 we describe extra supersymmetries and their algebras for spinning spaces. In Subsection 3.1 we apply the formalism for the motion of a spinning particle in the Schwarzschild–de Sitter spacetime. We discuss specific solutions and derive an exact equation for the precession of the perihelion of planar orbits in Subsection 3.2. In Subsection 3.3 we construct a new kind of supersymmetry generated by

Killing–Yano tensors of second-rank in the Schwarzschild–de Sitter spacetime. Finally, in Section 4 we present our remarks.

2. MOTIONS OF SPINNING PARTICLES IN CURVED SPACETIME

2.1. Spinning Space

A spinning space is an extension of an ordinary Riemannian manifold parametrized by local coordinates $\{x^\mu\}$, to a graded manifold parametrized by local coordiantes $\{x^\mu, \psi^\mu\}$, with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting) (Balachandran *et al.*, 1977; Barducci *et al.*, 1976, 1977, 1981a,b; Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Gibbons *et al.*, 1993; Gitman, 1996; Guth, 1981; Rietdijk, 1992; Turner, 1995, 1998; van Holten, 1991, 1992, 1994, 1995). This extension generates a supersymmetry in spinning space, which acts on the coordinates as

$$\delta x^\mu = -i\epsilon\psi^\mu, \quad \delta\psi^\mu = \epsilon\dot{x}^\mu, \tag{1}$$

where the dot denotes a derivative with respect to proper time and the infinitesimal parameter ϵ of the transformation is Grassmann-odd. The equations for extremal trajectories (*geodesics*) of spinning space describe the pseudoclassical mechanics of a Dirac fermion. To define the extremal trajectories we consider the supersymmetric action

$$S = \int_1^2 d\tau\left(\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right), \tag{2}$$

where the covariant derivative of ψ^μ is given by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda\Gamma^\mu_{\lambda\nu}\psi^\nu. \tag{3}$$

The variation of the action under arbitrary variations $(\delta x^\mu, \delta\psi^\mu)$ is

$$\begin{aligned}\delta S = \int_1^2 d\tau\Bigg\{&-\delta x^\mu\left(g_{\mu\nu}\frac{D^2x^\nu}{D\tau^2} + \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}\dot{x}^\nu\right)\\ &+i\Delta\psi^\mu g_{\mu\nu}\frac{D\psi^\nu}{D\tau} + \frac{d}{d\tau}\left(\delta x^\mu p_\mu - \frac{i}{2}\delta\psi^\mu g_{\mu\nu}\psi^\nu\right)\Bigg\},\end{aligned} \tag{4}$$

where p_μ is the canonical momentum:

$$p_\mu = g_{\mu\nu}\dot{x}^\nu - \frac{1}{2}i\Gamma_{\mu\kappa\lambda}\psi^\kappa\psi^\lambda, \tag{5}$$

and $R_{\kappa\lambda\mu\nu}$ is the Riemann curvature tensor. Moreover,

$$\Delta\psi^\mu = \delta\psi^\mu + \delta x^\lambda\Gamma^\mu_{\lambda\nu}\psi^\nu \tag{6}$$

is the covariantized variation of ψ^μ.

The trajectories, which make the action stationary under arbitrary variations δx^μ and $\delta\psi^\mu$ vanishing at the end points, are given by

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\dot{x}^\lambda\dot{x}^\nu = -\frac{1}{2}i\psi^\kappa\psi^\lambda R_{\kappa\lambda}{}^\mu{}_\nu\dot{x}^\nu, \tag{7}$$

$$\frac{D\psi^\mu}{D\tau} = 0. \tag{8}$$

Clearly, when $\psi^\mu = 0$, the solutions for $x^\mu(\tau)$ are ordinary geodesics in the bosonic submanifold.

The anticommuting spin variables are related to the standard antisymmetric spin tensor by

$$S^{\mu\nu} = -i\psi^\mu\psi^\nu, \tag{9}$$

and correspondingly Eqs. (7) and (8) describe the classical motion of a Dirac particle. Equation (7) implies the existence of a spin-dependent gravitational force (van Holten, 1991, 1992)

$$\frac{D^2 x^\mu}{D\tau^2} = \frac{1}{2}S^{\kappa\lambda}R_{\kappa\lambda}{}^\mu{}_\nu\dot{x}^\nu, \tag{10}$$

which is similar to the electromagnetic Lorentz force

$$\ddot{x}^\mu = \left(\frac{q}{m}\right)F^\mu{}_\nu\dot{x}^\nu, \tag{11}$$

with spin replacing the scalar electric charge (Khriplovich, 1989; van Holten, 1991, 1992) (here for unit mass). Equation (8) asserts that the spin is covariantly constant:

$$\frac{DS^{\mu\nu}}{D\tau} = 0. \tag{12}$$

The interpretation of $S^{\mu\nu}$ as spin tensor is corroborated by studying electromagnetic interactions of the particle (Barducci *et al.*, 1981a; Berezin and Marinov, 1975, 1977; Brink *et al.*, 1977; van Holten, 1991, 1992). From such an analysis it results that the spacelike components S^{ij} are proportional to the particle's magnetic dipole moment, while the time-like components S^{io} represent the electric dipole moment. The requirement that for free Dirac particles like free electrons and quarks the electric dipole moment vanishes in the rest frame can be written as a covariant constraint (Rietdijk, 1992)

$$g_{\nu\lambda}(x)S^{\mu\nu}\dot{x}^\lambda = 0, \tag{13}$$

which, in terms of the Grassmann coordinates, is equivalent to

$$g_{\mu\nu}(x)\dot{x}^\mu\psi^\nu = 0. \tag{14}$$

2.2. Symmetries and Generalized Killing Equations

We now look for specific variations δx^μ and $\Delta\psi^\mu$ of the form

$$\delta x^\mu = \mathcal{R}^\mu(x, \dot{x}, \psi) = R^{(1)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} R^{(n+1)_\mu}_{\nu_1 \cdots \nu_n}(x, \psi),$$

$$\delta\psi^\mu = \mathcal{S}^\mu(x, \dot{x}, \psi) = S^{(0)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} S^{(n)_\mu}_{\nu_1 \cdots \nu_n}(x, \psi), \qquad (15)$$

which leave the action off-shell invariant modulo boundary terms. If the Lagrangian transforms into a total derivative

$$\delta S = \int_1^2 d\tau \frac{d}{d\tau}\left(\delta x^\mu p_\mu - \frac{i}{2}\delta\psi^\mu g_{\mu\nu}\psi^\nu - \mathcal{J}(x, \dot{x}, \psi)\right), \qquad (16)$$

it follows that

$$\frac{d\mathcal{J}}{d\tau} = \mathcal{R}^\mu\left(g_{\mu\nu}\frac{D^2 x^\nu}{D\tau^2} + \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}\dot{x}^\nu\right) + i\mathcal{S}^\mu g_{\mu\nu}\frac{D\psi^\nu}{D\tau}. \qquad (17)$$

The right-hand side vanishes, if the equations of motion are satisfied, and then $\mathcal{J}$ is conserved. This is Noether's theorem. Otherwise, expanding $\mathcal{J}(x, \dot{x}, \psi)$ in terms of the four-velocity,

$$\mathcal{J}(x, \dot{x}, \psi) = J^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_n} J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \qquad (18)$$

and comparing the left- and right-hand sides of Eq. (17) with the ansatz (15) for δx^μ and $\Delta\psi^\mu$, one finds the following indentities:

$$J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi) = R^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \quad n \geq 1, \qquad (19)$$

$$S^{(n)}_{\mu_1 \cdots \mu_n \nu}(x, \psi) = i\frac{\partial J^{(n)}_{\mu_1 \cdots \mu_n}}{\partial \psi^\nu}(x, \psi), \quad n \geq 0. \qquad (20)$$

These equations have to satisfy a generalization of the Killing equation of the form (Gibbons *et al.*, 1993; Rietdijk, 1992; Rietdijk and van Holten, 1990)

$$J^{(n)}_{(\mu_1 \cdots \mu_n; \mu_{n+1})} + \frac{\partial J^{(n)}_{(\mu_1 \cdots \mu_n)}}{\partial \psi^\sigma}\Gamma_{(\mu_{n+1})\kappa}{}^{\sigma}\psi^\kappa = \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\nu(\mu_{n+1})} J^{(n+1)\nu}_{(\mu_1 \cdots \mu_n)}, \qquad (21)$$

where the parentheses denote symmetrization with norm one over the indices enclosed. Writing $R^{(1)}_\mu = R_\mu$, $R^{(2)}_{\mu\nu} = K_{\mu\nu}$, $L^{(3)}_{\mu\nu\lambda} = L_{\mu\nu\lambda}$, etc., and $J^{(0)} = B$, this reduces for the lowest components to

$$B_{,\mu} + \frac{\partial B}{\partial \psi^\sigma}\Gamma_{\mu\kappa}{}^{\sigma}\psi^\kappa = \frac{i}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\kappa\mu}R^\kappa, \qquad (22)$$

$$R_{(\mu;\nu)} + \frac{\partial R_{(\mu}}{\partial \psi^\sigma} \Gamma_{\nu)\kappa}^{\ \sigma} \psi^\kappa = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\kappa(\mu} K_{\nu)}^{\ \kappa}, \tag{23}$$

$$K_{(\mu\nu;\lambda)} + \frac{\partial K_{(\mu\nu}}{\partial \psi^\sigma} \Gamma_{\lambda)\kappa}^{\ \sigma} \psi^\kappa = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\kappa(\mu} L_{\nu\lambda)}^{\ \ \kappa}, \text{ etc.} \tag{24}$$

These equations hold independently of the equations of motion.

In the scalar case, neglecting the Grassmann variables $\{\psi^\mu\}$, all the generalized Killing equations (21) are homogeneous and decoupled. The first equation ($n = 0$) shows that $J^{(0)} = B$ is a trivial constant, the next one ($n = 1$) is the equation for the Killing vectors $J^{(1)}_\mu = R_\mu$ and so on. In general, for a given n, neglecting all spin variables, Eq. (21) defines a Killing tensor of valence n:

$$J^{(n)}_{(\mu_1\cdots\mu_n;\mu_{n+1})}(x) = 0, \tag{25}$$

and from Eq. (18)

$$\mathcal{J} = J^{(n)}_{\mu_1\cdots\mu_n}(x)\dot{x}^{\mu_1}\cdots\dot{x}^{\mu_n} \tag{26}$$

is a first integral of the geodesic equation (Dietz and Rüdinger, 1981).

2.3. Generic Solutions for Spinning Space

In contrast to the scalar particle, the spinning particle admits several conserved quantities of motion in a curved spacetime with metric $g_{\mu\nu}(x)$ (Rietdijk and van Holten, 1990). Specifically, there are four independent *generic* constants of motion that exist in any theory. These are as follows:

1. Similar to the bosonic case $g_{\mu\nu}$ itself is a Killing tensor:

$$K_{\mu\nu} = g_{\mu\nu}, \tag{27}$$

with all other Killing vectors and tensors (bosonic as well as fermionic) equal to zero. The corresponding constant of motion is the world-line Hamiltonian,

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu, \tag{28}$$

where

$$\Pi_\mu = g_{\mu\nu}\dot{x}^\nu \tag{29}$$

is the covariant momentum.

2. The Grassmann-odd Killing vectors

$$R^\mu = \psi^\mu, \quad T^\nu_\mu = i\delta^\nu_\mu \tag{30}$$

provide another obvious solution. Here again all other Killing vectors and tensors are taken to vanish. This solution gives the supercharge

$$Q = \Pi_\mu \psi^\mu. \tag{31}$$

3. The spinning particle action has a second nonlinear supersymmetry generated by Killing vectors

$$R_\mu = \frac{-i^{[d/2]}}{(d-1)!}\sqrt{-g}\,\varepsilon_{\mu\nu_1\cdots\nu_{d-1}}\psi^{\nu_1}\cdots\psi^{\nu_{d-1}},$$
$$T_{\mu\nu} = \frac{i^{[(d-2)/2]}}{(d-2)!}\sqrt{-g}\,\varepsilon_{\mu\nu\nu_1\cdots\nu_{d-2}}\psi^{\nu_1}\cdots\psi^{\nu_{d-2}}. \tag{32}$$

Obviously, the Grassmann parities of $(R_\mu, T_{\mu\nu})$ depend on d, the number of spacetime dimensions. The corresponding constant of motion is the dual supercharge

$$Q^* = \frac{-i^{[d/2]}}{(d-1)!}\sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_d}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d}. \tag{33}$$

4. Finally, there is a nontrivial Killing scalar

$$\Gamma_* = J^{(0)} = \frac{-i^{[d/2]}}{d!}\sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_d}\psi^{\mu_1}\cdots\psi^{\mu_d}, \tag{34}$$

which acts as the Hodge star duality operator on ψ^μ. In quantum mechanics it becomes the γ^{d+1} element of the Dirac algebra. Because of this reason Γ_* is referred to as the chiral charge.

The Poisson–Dirac bracket for functions of the covariant phase-space variables (x, Π, ψ) is defined by

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu}\mathcal{D}_\mu G - \mathcal{R}_{\mu\nu}\frac{\partial F}{\partial \Pi_\mu}\frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^\mu}\frac{\partial G}{\partial \psi_\mu}, \tag{35}$$

where

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma_{\mu\nu}^\lambda \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} - \Gamma_{\mu\nu}^\lambda \psi^\nu \frac{\partial F}{\partial \psi^\lambda}, \quad \mathcal{R}_{\mu\nu} = \frac{i}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\mu\nu}, \tag{36}$$

and a_F is the Grassmann parity of $F : a_F = (0, 1)$ for $F =$ (even, odd). Using this bracket one finds that

$$\{Q, Q\} = -2iH, \quad \{Q, \Gamma_*\} = -iQ^*. \tag{37}$$

Clearly, $d = 2$ is an exceptional case: Q^* is linear and acts as an ordinary supersymmetry:

$$\{Q^*, Q^*\} = -2iH, \quad \{Q^*, \Gamma_*\} = -iQ. \tag{38}$$

It implies that in two dimensions the theory actually possesses an $N = 2$ supersymmetry. For $d \neq 2$, the right-hand side of Eq. (38) vanishes.

2.4. Nongeneric Solutions for Spinning Space

The appearance of nongeneric supersymmetries for the spinning particle in curved spacetime depends on the specific form of the metric $g_{\mu\nu}(x)$. More explicitly, the existence of Killing–Yano tensors is both necessary and sufficient for the appearance of a new supersymmetry for the spinning space (Gibbons *et al.*, 1993; van Holten, 1994, 1995).

We remind that a tensor $f_{\mu\nu}$ is called a Killing–Yano tensor of valence 2 (Dietz and Rüdinger, 1981; Yano, 1952) if and only if it is completely antisymmetric and it satisfies the Penrose–Floyd equation (Floyd, 1973; Penrose, 1973)

$$D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} = 0. \tag{39}$$

The Stackel–Killing tensor $K_{\mu\nu}$, which is the solution of (24), has a certain square root (Floyd, 1973; Penrose, 1973) such that

$$K_\mu^\nu = f_\mu^a f_a^\nu. \tag{40}$$

Using the vielbein (tetrad) $e_\mu^a(x)$ the double vector f_μ^a can be written as follows:

$$f_\mu^a = f_{\mu\nu} e^{\nu a}. \tag{41}$$

One now finds that the theory admits nongeneric supersymmetries of the type

$$\delta x^\mu = -i\epsilon f^\mu{}_a \psi^a \equiv -i\epsilon J^{(1)\mu}. \tag{42}$$

Such a transformation is generated by a phase-space function $Q_{\rm f}$:

$$Q_{\rm f} = J^{(1)\mu}\Pi_\mu + J^{(0)}, \tag{43}$$

where $J^{(0)}(x, \psi)$ and $J^{(1)}(x, \psi)$ are independent of Π. When this ansatz is inserted into the generalized Killing equations (21) with $n = 0$, it follows that (Gibbons *et al.*, 1993)

$$J^{(0)} = \frac{i}{3!} c_{abc}(x)\psi^a\psi^b\psi^c, \tag{44}$$

where the tensor c_{abc} is

$$c_{abc} = -2D_{[a} f_{bc]} \equiv -2e^\mu{}_a e^\nu{}_b e^\lambda{}_c D_{[\mu} f_{\nu\lambda]}. \tag{45}$$

Here, the square brackets denote antisymmetrization with norm one over the indices enclosed. Let there be N such symmetries specified by N sets of tensors (f_{ia}^μ, c_{iabc}), $i = 1, \ldots, N$. The corresponding generators will be

$$Q_i = f_{ia}^\mu \Pi_\mu \psi^a + \frac{i}{3!} c_{iabc}(x)\psi^a\psi^b\psi^c. \tag{46}$$

Obviously, for $f^{\mu}{}_{a} = e^{\mu}{}_{a}$ and $c_{abc} = 0$, the supercharge (31) is precisely of this form. It is therefore convenient to assign the index $i = 0 : Q = Q_0$, $e^{\mu}{}_{a} = f^{\mu}_{0a}$, etc., when we refer to the quantities defining the standard supersymmetry.

The Poisson–Dirac bracket (35) gives the following algebra for the conserved charges Q_i:

$$\{Q_i, Q_j\} = -2i Z_{ij}, \tag{47}$$

where

$$Z_{ij} = \frac{1}{2} K^{\mu\nu}_{ij} \Pi_\mu \Pi_\nu + I^{\mu}_{ij} \Pi_\mu + G_{ij}, \tag{48}$$

and

$$K^{\mu\nu}_{ij} = \frac{1}{2}\left(f^{\mu}_{ia} f^{\nu a}_{j} + f^{\nu}_{ia} f^{\mu a}_{j}\right), \tag{49}$$

$$\begin{aligned} I^{\mu}_{ij} &= \frac{1}{2} i \psi^a \psi^b I^{\mu}_{ijab} \\ &= \frac{1}{2} i \psi^a \psi^b \left(f^{\nu}_{ib} D_\nu f^{\mu}_{ja} + f^{\nu}_{jb} D_\nu f^{\mu}_{ia} + \frac{1}{2} f^{\mu c}_{i} c_{jabc} + \frac{1}{2} f^{\mu c}_{j} c_{iabc}\right), \end{aligned} \tag{50}$$

$$\begin{aligned} G_{ij} &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d G_{ijabcd} \\ &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f^{\mu}_{ic} f^{\nu}_{jd} + \frac{1}{2} c_{iab}{}^{e} c_{jcde}\right). \end{aligned} \tag{51}$$

The functions Z_{ij} satisfy the generalized Killing equations. Hence their bracket with the Hamiltonian vanishes and they are constants of motion:

$$\frac{dZ_{ij}}{d\tau} = 0. \tag{52}$$

For $i = j = 0$, (47) reduces to the usual supersymmetry algebra

$$\{Q, Q\} = -2i H. \tag{53}$$

If i or j is not equal to zero, Z_{ij} correspond to new bosonic symmetries, unless $K^{\mu\nu}_{ij} = \lambda_{(ij)} g^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (may be zero). In that case the corresponding Killing vector I^{μ}_{ij} and scalar G_{ij} disappear identically. Further, the supercharges for $\lambda_{(ij)} \neq 0$ close on the Hamiltonian. This shows the existence of a second supersymmetry of the standard type. Thus the theory admits an N-extended supersymmetry with $N \geq 2$. On the contrary, if there exists a second independent Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, there exists a genuine new type of supersymmetry.

The quantity Q_i is a superinvariant, that is,

$$\{Q_i, Q\} = 0 \tag{54}$$

for the bracket (35), if and only if

$$K_{0i}^{\mu\nu} = f^{\mu}{}_{a}e^{\nu a} + f^{\nu}{}_{a}e^{\mu a} = 0. \tag{55}$$

In this case, the full constant of motion $Z_{\rm ij}$ can be constructed directly by repeated differentiation of f_a^{μ} (Gibbons *et al.*, 1993).

As the Z_{ij} are symmetric in (ij) we can diagonalize them. This provides the algebra

$$\{Q_i, Q_j\} = -2i\delta_{ij}Z_i, \tag{56}$$

with $N+1$ conserved bosonic charges Z_i. If all Q_i satisfy condition (55), the first of these diagonal charges (with $i = 0$) is the Hamiltonian: $Z_0 = H$.

3. SPINNING PARTICLES IN SCHWARZSCHILD–DE SITTER SPACETIME

3.1. Laws of Motion in Schwarzschild–de Sitter Spacetime

As an application of the generalized Killing equations of spinning space we investigate the motion of a spinning particle in the Schwarzschild–de Sitter spacetime described by the metric

$$ds^2 = -Vdt^2 + \frac{dr^2}{V} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{57}$$

where

$$V(r) = 1 - \frac{\alpha}{r} - \frac{1}{3}\Lambda r^2, \tag{58}$$

with $\alpha = 2M$, M being the total mass of the gravitating body and Λ the cosmological parameter. This metric possesses four Killing vector fields of the form

$$D^{(a)} \equiv R^{(a)\mu}\partial_\mu, \quad a = 0, \ldots, 3, \tag{59}$$

where

$$D^{(0)} = \frac{\partial}{\partial t}, \quad D^{(1)} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},$$
$$D^{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \quad D^{(3)} = \frac{\partial}{\partial\varphi}. \tag{60}$$

These Killing vector fields describe the time-translation invariance and the spatial rotation symmetry of the gravitating field. They generate the Lie algebra $o(1,1) \times so(3)$:

$$[D^{(a)}, D^{(b)}] = -\varepsilon^{abc}D^{(c)}, \quad [D^{(0)}, D^{(a)}] = 0, \quad (a, b, c = 1, 2, 3). \tag{61}$$

The first generalized Killing equation (22) shows that for each Killing vector $R_\mu^{(a)}$ there is an associated Killing scalar $B^{(a)}$. So, if we limit ourselves to variations (15) that terminate after the terms linear in $\dot{x}^\mu$, we get the constants of motion

$$J^{(a)} = B^{(a)} + m\dot{x}^\mu R_\mu^{(a)}, \tag{62}$$

which asserts that the contribution of spin is contained in the Killing scalars $B^{(a)}$. Without the Killing scalars the Killing vector itself does not give a conserved quantity of motion.

Solving Eq. (22) for the Killing scalars of the Schwarzschild–de Sitter spacetime we obtain

$$\begin{aligned}
B^{(0)} &= \left(\frac{\alpha}{2r^2} - \frac{1}{3}\Lambda r\right) S^{tr},\\
B^{(1)} &= -r\sin\varphi S^{r\theta} - r\sin\theta\cos\theta\cos\varphi S^{r\varphi} + r^2\sin^2\theta\cos\varphi S^{\theta\varphi},\\
B^{(2)} &= r\cos\varphi S^{r\theta} - r\sin\theta\cos\theta\sin\varphi S^{r\varphi} + r^2\sin^2\theta\sin\varphi S^{\theta\varphi},\\
B^{(3)} &= r\sin^2\theta S^{r\varphi} + r^2\sin\theta\cos\theta S^{\theta\varphi},
\end{aligned} \tag{63}$$

where the spin–tensor notation introduced in Eq. (9) has been used. From Eq. (62) the four conserved quantities $J^{(a)}$ are found as follows:

$$\begin{aligned}
J^{(0)} \equiv E &= m\left(1 - \frac{\alpha}{r} - \frac{1}{3}\Lambda r^2\right)\frac{dt}{d\tau} - \left(\frac{\alpha}{2r^2} - \frac{1}{3}\Lambda r\right) S^{rt},\\
J^{(1)} &= -r\sin\varphi\left(mr\frac{d\theta}{d\tau} + S^{r\theta}\right) - \cos\varphi\big(\cot\theta\, J^{(3)} - r^2 S^{\theta\varphi}\big),\\
J^{(2)} &= r\cos\varphi\left(mr\frac{d\theta}{d\tau} + S^{r\theta}\right) - \sin\varphi\big(\cot\theta\, J^{(3)} - r^2 S^{\theta\varphi}\big),\\
J^{(3)} &= r\sin^2\theta\left(mr\frac{d\varphi}{d\tau} + S^{r\varphi}\right) + r^2\sin\theta\cos\theta S^{\theta\varphi}\big),
\end{aligned} \tag{64}$$

In addition to these conserved quantities, there are four generic constants of motion as described in Subsection 2.3. We consider motion for which

$$H = -\frac{m^2}{2}. \tag{65}$$

This yields geodesic motion:

$$g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2. \tag{66}$$

The condition (13) gives the supersymmetric constraint

$$Q = 0, \tag{67}$$

which enables one to solve for ψ^t in terms of the spatial components ψ^i:

$$\left(1-\frac{\alpha}{r}-\frac{1}{3}\Lambda r^2\right)\frac{dt}{d\tau}\psi^t = \frac{1}{(1-\alpha/r-\Lambda r^2/3)}\frac{dr}{d\tau}\psi^r + r^2\left(\frac{d\theta}{d\tau}\psi^\theta + \sin^2\theta\frac{d\varphi}{d\tau}\psi^\varphi\right). \tag{68}$$

As a result, the dual supercharge Q^* and the chiral charge Γ_* vanish as well: $Q^* = \Gamma_* = 0$.

From (64) we can derive a useful identity

$$r^2\sin\theta S^{\theta\varphi} = J^{(1)}\sin\theta\cos\varphi + J^{(2)}\sin\theta\cos\varphi + J^{(3)}\cos\theta, \tag{69}$$

which, in physical terms, simply states that there is no orbital angular momentum in the radial direction.

Combining Eqs. (64)–(67), one can obtain a complete set of first integrals of motion as follows:

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{1}{(1-\alpha/r-\Lambda r^2/3)}\left[\frac{E}{m}+\frac{1}{m}\left(\frac{\alpha}{2r^2}-\frac{\Lambda r}{3}\right)S^{rt}\right],\\
\frac{dr}{d\tau} &= \left\{\left(1-\frac{\alpha}{r}-\frac{1}{3}\Lambda r^2\right)^2\left(\frac{dt}{d\tau}\right)^2 - 1 + \frac{\alpha}{r}+\frac{1}{3}\Lambda r^2\right.\\
&\quad\left. - r^2\left(1-\frac{\alpha}{r}-\frac{1}{3}\Lambda r^2\right)\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]\right\}^{1/2},\\
\frac{d\theta}{d\tau} &= \frac{1}{mr^2}\left(-J^{(1)}\sin\varphi + J^{(2)}\cos\varphi - rS^{r\theta}\right),\\
\frac{d\varphi}{d\tau} &= \frac{1}{mr^2\sin^2\theta}J^{(3)} - \frac{1}{mr}S^{r\varphi} - \frac{1}{m}\cot\theta S^{\theta\varphi},
\end{aligned} \tag{70}$$

where

$$S^{rt} = \frac{mr^2}{E}\left(\frac{d\theta}{d\tau}S^{r\theta}+\sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right). \tag{71}$$

Finally Eq. (8) with Eq. (9) gives equations for the spin. The equations that are left for solution are

$$\begin{aligned}
\frac{dS^{r\theta}}{d\tau} &= -\frac{1}{r}\frac{dr}{d\tau}S^{r\theta} + \sin\theta\cos\theta\frac{d\varphi}{d\tau}S^{r\varphi} - r\sin^2\theta\left(1-\frac{3\alpha}{2r}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi},\\
\frac{dS^{r\varphi}}{d\tau} &= \cot\theta\frac{d\varphi}{d\tau}S^{r\theta} - \left(\frac{1}{r}\frac{dr}{d\tau}+\cot\theta\frac{d\theta}{d\tau}\right)S^{r\varphi} + r\left(1-\frac{3\alpha}{2r}\right)\frac{d\theta}{d\tau}S^{\theta\varphi},
\end{aligned} \tag{72}$$

where $S^{\theta\varphi}$ is given by (69).

For $\Lambda = 0$ the above equations reduce to the Schwarzschild results (Rietdijk and van Holten, 1993). One has to solve Eqs. (70)–(72) to obtain the full solution of the equations of motion for all coordinates and spins.

3.2. Special Solutions

We apply the results obtained in Subsection 3.1 to study the special case of motion in a plane, for which we choose $\theta = \pi/2$. In contrast to scalar particles, this is not the generic case, because in general orbital angular momentum is not conserved separately.

Planar motion for spinning particles is strictly possible only in special cases, in which orbital and spin angular momentum are separately conserved. This happens only in two kinds of situations: the orbital angular momentum vanishes, or spin and orbital angular momentum are parallel.

With $\theta = \pi/2$ and $\dot{\theta} = 0$, Eqs. (70)–(72) become

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{1}{(1 - \alpha/r - \Lambda r^2/3)}\left[\frac{E}{m} + \frac{1}{E}\left(\frac{\alpha}{2} - \frac{\Lambda}{3}r^3\right)\frac{d\varphi}{d\tau}S^{r\varphi}\right],\\
\frac{dr}{d\tau} &= \left\{\left(1 - \frac{\alpha}{r} - \frac{\Lambda}{3}r^2\right)^2\left(\frac{dt}{d\tau}\right)^2 - 1 + \frac{\alpha}{r} + \frac{\Lambda}{3}r^2\right.\\
&\qquad \left. - r^2\left(1 - \frac{\alpha}{r} - \frac{\Lambda}{3}r^2\right)\left(\frac{d\varphi}{d\tau}\right)^2\right\}^{1/2},\\
\frac{d\varphi}{d\tau} &= \frac{1}{mr^2}J^{(3)} - \frac{1}{mr}S^{r\varphi},\\
\frac{d}{d\tau}(rS^{r\theta}) &= -r^2\left(1 - \frac{3\alpha}{2r}\right)S^{\theta\varphi}\frac{d\varphi}{d\tau},\\
\frac{d}{d\tau}(rS^{r\varphi}) &= 0.
\end{aligned} \tag{73}$$

The third and the last equations of (73) express the fact that the orbital angular momentum and the component of the spin perpendicular to the plane in which the particle moves are separately conserved:

$$rS^{r\varphi} \equiv \Sigma, \quad mr^2\dot{\varphi} = J^{(3)} - \Sigma \equiv L, \tag{74}$$

where Σ and L are two constants. From the first of Eq. (73) we find a formula for the gravitational redshift in the form

$$dt = \frac{d\tau}{(1 - \alpha/r - \Lambda r^2/3)}\left[\frac{E}{m} + \frac{1}{mEr^3}\left(\frac{\alpha}{2} - \frac{\Lambda}{3}r^3\right)L\Sigma\right]. \tag{75}$$

For nonzero orbital angular momentum L, the time-dilation receives a contribution from spin–orbit coupling. Thus we see that time-dilation is not a purely geometric effect, but also has a dynamical component (van Holten, 1991, 1992).

From Eqs. (69), third of (70), and fourth of (73) with $\theta = \pi/2$, we find that there are indeed only two possibilities:

$$\text{(i)}\ \dot{\varphi} = 0, \quad \text{(ii)}\ S^{\theta\varphi} = 0. \tag{76}$$

Case(i). $\dot{\varphi} = 0$ implies that $L = 0$. The particle moves along a fixed radius. The equation of motion of the particle for a distant observer is described by

$$\frac{dr}{dt} = \left(1 - \frac{\alpha}{r} - \frac{1}{3}\Lambda r^2\right)\sqrt{1 - \frac{m^2}{E^2}\left(1 - \frac{\alpha}{r} - \frac{1}{3}\Lambda r^2\right)} \tag{77}$$

as in the case of a spinless particle. If we choose $\varphi = 0$ for the path of the particle, then the spin tensor components are all conserved:

$$r^2 S^{\theta\varphi} = J^{(1)}, \quad r S^{r\theta} = J^{(2)}, \quad r S^{r\varphi} = J^{(3)}. \tag{78}$$

Case(ii). $\dot{\varphi} \neq 0$ implies that

$$S^{\theta\varphi} = 0, \quad S^{r\theta} = 0, \quad J^{(1)} = J^{(2)} = 0. \tag{79}$$

This states that the spin is parallel to the orbital angular momentum. From Eq. (73) for $\dot{r}$ and $\dot{\varphi}$ we obtain following equation for the orbit of the particle:

$$\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 = \frac{E^2 - m^2}{L^2}r^2 - 1 + \frac{m}{L}\left(\alpha - \frac{2}{3}\Lambda r^3\right) \times \left(\frac{mr}{L} + \frac{J^{(3)}}{mr}\right) + \Lambda r^2\left(1 + \frac{m^2 r^2}{L^2}\right). \tag{80}$$

In terms of dimensionless variables

$$\epsilon = \frac{E}{m}, \quad x = \frac{r}{\alpha}, \quad \ell = \frac{L}{m\alpha}, \quad \Delta = \frac{\Sigma}{L}, \lambda = \alpha\sqrt{|\Lambda|}, \tag{81}$$

Eq. (80) takes the form

$$\frac{\ell^2}{x^4}\left(\frac{dx}{d\varphi}\right)^2 = \alpha^2\dot{x}^2 = \epsilon^2 - U_R(x, \ell^2, \lambda^2), \tag{82}$$

where

$$U_R(x, \ell^2, \lambda^2) = 1 + \left(\frac{2}{3}(1 + \Delta)\ell^2 - x^3 + \frac{2}{3}x^2 - \ell^2 x\right)\lambda^2 - \frac{1}{x} + \frac{\ell^2}{x^2} - \ell^2(1 + \Delta)\frac{1}{x^3} \tag{83}$$

defines an effective potential.

All calculations presented here are rather formal, as Δ is not a pure number. For realistic physics situations, we have to replace Δ in certain limiting cases by a real number. As mentioned in Introduction section this may be done by averaging Δ over some suitable density, as in the path-integral. In the following we assume that such an averaging procedure has been performed, and treat Δ as a classical variable. To avoid any inconsistency that may result from this semiclassical approximation, we presuppose the numerical value of Δ to be small: $\Delta \ll 1$.

For bound state orbits it is necessary that $\epsilon < 1$. The function $U_R(x, \ell^2, \lambda^2)$ has a point of inflection that corresponds to a circular orbit with minimum radius given by

$$\lambda^2[2(3x-1)x+\ell^2]x^3+3\ell^2(1+\Delta)\frac{1}{x}-\ell^2=0, \tag{84}$$

where

$$\ell^2=\frac{1}{P}\left[x-3\lambda^2\left(1-\frac{4}{9x}\right)x^5\right],$$
$$P=2-3(1+\Delta)\frac{1}{x}+\lambda^2x^3, \tag{85}$$

with $P > 0$. For this critical orbit, the energy is given by

$$\epsilon_{\text{crit}}^2=1-\frac{1}{x}-\left(x^3-\frac{2x^2}{3}\right)\lambda^2+\ell^2R\frac{1}{x^2}, \tag{86}$$

with

$$R=1-(1+\Delta)\frac{1}{x}+\left(\frac{2}{3}(1+\Delta)x^2-x^3\right)\lambda^2,$$

and the time-dilation factor is expressed by

$$\left(\frac{dt}{d\tau}\right)_{\text{crit}}=\frac{1}{N}\left[\epsilon_{\text{crit}}-\frac{\ell^2\Delta}{2\epsilon_{\text{crit}}x^3}(1-2(1-N)x)\right], \tag{87}$$

with

$$N=1-\frac{1}{x}+\frac{1}{3}\lambda^2x^2.$$

Equation (84) with $\lambda = 0$ gives the radius of the minimal circular orbit as

$$x=\ell^2=3(1+\Delta). \tag{88}$$

The energy and the time-dilation for this orbit are respectively given by

$$\epsilon_{\text{crit}}=\frac{\sqrt{2}}{3}\left(2+\frac{\Delta}{8}\right),\quad \left(\frac{dt}{d\tau}\right)_{\text{crit}}=\sqrt{2}\left(1-\frac{3}{8}\Delta\right) \tag{89}$$

to first order in Δ. This is the Schwarzschild result as obtained in Rietdijk and van Holten (1993).

The orbits of the particle that approach precessing ellipses (because of relativistic effects) are described by

$$x = \frac{\kappa}{1 + \varepsilon \cos[\varphi - w(\varphi)]}, \tag{90}$$

where $\kappa = \kappa/\alpha$, κ being the semilatus rectum and ε is the eccentricity with $0 < \varepsilon < 1$. The perihelion and aphelion are given by

$$\varphi_{\text{ph}}^{(t)} - w\left(\varphi_{\text{ph}}^{(t)}\right) = 2t\pi, \quad \varphi_{\text{ah}}^{(t)} - w\left(\varphi_{\text{ah}}^{(t)}\right) = (2t+1)\pi. \tag{91}$$

The angle $\varphi_{\text{ph}}^{(t)}$ is the tth perihelion of the particle, while $w(\varphi_{\text{ph}}^{(t)})$ is the amount of precession of the perihelion after t revolutions. Hence the precesssion of the perihelion after one revolution is

$$\Delta w \equiv w\left(\varphi_{\text{ph}}^{(1)}\right) - w\left(\varphi_{\text{ph}}^{(0)}\right) = \varphi_{\text{ph}}^{(1)} - \varphi_{\text{ph}}^{(0)} - 2\pi \equiv \Delta\varphi - 2\pi. \tag{92}$$

The energy at the perihelion/aphelion is given by

$$\begin{aligned}\epsilon^2 = 1 &- \left(\frac{1 \pm \varepsilon}{\kappa}\right) + \ell^2\left(\frac{1 \pm \varepsilon}{\kappa}\right)^2 - \ell^2(1+\Delta)\left(\frac{1 \pm \varepsilon}{\kappa}\right)^3 \\ &+ \lambda^2\left[\frac{2}{3}\ell^2(1+\Delta) - \left(\frac{\kappa}{1 \pm \varepsilon}\right)^3 + \frac{2}{3}\left(\frac{\kappa}{1 \pm \varepsilon}\right)^2 - \ell^2\left(\frac{\kappa}{1 \pm \varepsilon}\right)\right].\end{aligned} \tag{93}$$

Since the energy ϵ is a constant of motion, it follows from comparison of both expressions for ϵ^2 that

$$\ell^2 = \frac{\kappa^2}{3(1-\varepsilon^2)^2}\,\frac{3\kappa(1-\varepsilon^2)^3 + \lambda^2\kappa^4[4(1-\varepsilon^2) - 3(3+\varepsilon^2)\kappa]}{(1-\varepsilon^2)\kappa[2\kappa - (1+\Delta)(3+\varepsilon^2)] + \lambda^2\kappa^5}. \tag{94}$$

Using the above results and introducing

$$y = \varphi - w(\varphi) \tag{95}$$

the equation of motion (82) can be put in the form

$$d\varphi = \frac{\varepsilon}{\kappa}\,\frac{\sin y(1+\varepsilon\cos y)^{3/2}}{\left(\sum_{r=0}^{7} A_r(\varepsilon\cos y)^r\right)^{1/2}}dy, \tag{96}$$

where

$$\begin{aligned}A_0 = &\frac{1}{3\kappa^2\ell^2}[3\kappa^2(\epsilon^2-1) + 3(\kappa-\ell^2) + \lambda^2\kappa^2(3\kappa^2 - 2\kappa + 3\ell^2)] \\ &+ \frac{1}{3\kappa^2}(1+\Delta)(3 - 2\kappa^3\lambda^2),\end{aligned}$$

$$A_1 = \frac{1}{3\kappa^2\ell^2}[3\kappa(3\kappa(\epsilon^2 - 1) + 4) - 15\ell^2 - 2\kappa^3(\kappa - 3\ell^2)\lambda^2] + \frac{2}{3\kappa^3}(1+\Delta)(9 - \kappa^3\lambda^2),$$

$$A_2 = \frac{1}{\kappa^2\ell^2}[3\kappa(\kappa(\epsilon^2 - 1) + 2) - 10\ell^2 + \kappa^2\ell^2\lambda^2] + \frac{1}{\kappa^3}(1+\Delta)(15 - 2\kappa^3\lambda^2),$$

$$A_3 = \frac{1}{\kappa^2\ell^2}[\kappa(\kappa(\epsilon^2 - 1) + 4) - 10\ell^2] + \frac{2}{3\kappa^3}(1+\Delta)(30 - \kappa^3\lambda^2),$$

$$A_4 = \frac{1}{\kappa^2\ell^2}(\kappa - 5\ell^2) + \frac{15}{\kappa^3}(1+\Delta),$$

$$A_5 = -\frac{1}{\kappa^3}[\kappa - 6(1+\Delta)],$$

$$A_6 = \frac{1}{\kappa^3}(1+\Delta),$$

$$A_7 = \frac{1}{\kappa^4}[1 - 2(1+\Delta)],$$

ϵ^2 and ℓ^2 are given by (93) and (94). Then $\Delta\varphi$ as defined in (92) is obtained by integrating (96) from one perihelion to the next one with $0 \le y \le 2\pi$. The result gives

$$\Delta\varphi = \frac{2\pi}{a}\left[1 + \frac{1}{512}\varepsilon^2(15b + 70c) + \cdots\right], \tag{97}$$

where $a = \sqrt{A_0}\,\kappa/\varepsilon$, $b = A_1/A_0$, $c = A_2/A_0$. In evaluating this expression we should disregard terms of order Δ^2. With $\lambda = 0$, Eq. (97) reduces to

$$\Delta\varphi = 2\pi\left[1 + \frac{3}{2\kappa}(1+\Delta) + \frac{3}{16\kappa^2}(\varepsilon^2 + 18)(1+\Delta)^2 + \cdots\right], \tag{98}$$

which is exactly the result as derived in Rietdijk and van Holten (1993) for the Schwarzschild spacetime.

3.3. Nongeneric Supersymmetry

In this subsection, we apply the results of Subsection 2.4 to investigate a new type of supersymmetry in the Schwarzschild–de Sitter spacetime described by the metric (57). The Killing–Yano tensor $f_{\mu\nu}(x)$ for this spacetime is defined by

$$\frac{1}{2} f_{\mu\nu}dx^\mu \wedge dx^\nu = r\sin\theta d\theta \wedge r^2 d\varphi, \tag{99}$$

while the vierbein $e_\mu^a(x)$ is given by the following expressions:

$$e_\mu^0 dx^\mu = -U dt, \quad e_\mu^1 dx^\mu = U^{-1} dr, \quad e_\mu^2 dx^\mu = r d\theta, \quad e_\mu^3 dx^\mu = r \sin\theta d\varphi, \tag{100}$$

where $U \equiv \sqrt{V}$. The components of $f_\mu^a(x)$ can be written as follows:

$$f_\mu^0 dx^\mu = 0 = f_\mu^1 dx^\mu, \quad f_\mu^2 dx^\mu = -r^2 \sin\theta d\varphi, \quad f_\mu^3 dx^\mu = r^2 d\theta. \tag{101}$$

From Eq. (45) we obtain the components of $c_{abc}(x)$ as follows:

$$c_{012} = 0, \quad c_{013} = 0, \quad c_{023} = 0, \quad c_{123} = -2U. \tag{102}$$

Inserting the quantities derived in Eqs. (101) and (102) into Eq. (46) we obtain the new supersymmetry generator $Q_{\rm f}$ for the Schwarzschild–de Sitter spacetime. From Eqs. (49)–(51) the Killing tensor, vector, and scalar are constructed as follows:

$$K_{\mu\nu}(x) dx^\mu dx^\nu = r^4(\sin^2\theta d\varphi^2 + d\theta^2), \tag{103}$$

$$\begin{aligned} I_\mu(x) dx^\mu = {} & r^2[U(\sin\theta S^{r\varphi} + U\cos\theta S^{\theta\varphi}) \\ & - V\cos\theta S^{\theta\varphi}] d\varphi + r^2 U S^{r\theta} d\theta, \end{aligned} \tag{104}$$

$$G = 0. \tag{105}$$

From the Poisson–Dirac bracket (35) it can be verified straightforwardly that these equations satisfy the *so*(3,1) algebra.

The expression for $Q_{\rm f}$ and Eqs. (103)–(105) then define the conserved charge

$$Z = \frac{i}{2}\{Q_{\rm f}, Q_{\rm f}\}. \tag{106}$$

4. REMARKS

The spinning particle model is a world line supersymmetric extension of the ordinary relativistic point particle. It is a theory that describes in a pseudoclassical way a Dirac fermion moving in an arbitrary spacetime. Together with the usual spacetime coordinates, the model involves anticommuting vectorial coordinates that take into account the spin degrees of freedom. Along the world line of the particle there is a supersymmetry between the fermionic spin variables and the bosonic position coordinates. The model gives a one-dimensional supersymmetric field theory on the world line.

There is no classical interpretation for the anticommuting spin variables. One has to average over the spin variables to get the “observable” trajectories. Indeed, it is possible to quantize the model giving rise to supersymmetric quantum mechanics and then the conservation law for the supercharges becomes the Dirac equation.

Carter and McLenaghan (1979) showed that the Killing–Yano tensors play a key role in the Dirac theory on a curved spacetime. The study of the generalized

Killing equations strengthens the connection of the Killing–Yano tensors with the supersymmetric classical and quantum mechanics on a curved manifold.

The Stackel–Killing tensor determines a constant of motion (directly) for a scalar particle in a curved spacetime, whereas for a spinning particle it requires nontrivial contributions from spin. These spin-dependent portions are described by the Killing–Yano-type square root of the Stackel–Killing tensor. We have presented a detailed discussion on how to construct conserved quantities using Killing–Yano tensors for the Schwarzschild–de Sitter spacetime. This spacetime has an interesting property that it is asymptotically de Sitter instead of being asymptotically flat. We have solved the equations of motion for the case when θ is held fixed. Because of the presence of cosmological parameter the result of our study is interesting from the point of view of the inflationary scenario of the early universe. From our result one can get the result for the Schwarzschild spacetime as obtained in Rietdijk and van Holten (1993) by simply choosing the cosmological parameter $\Lambda = 0$. Besides, by neglecting the mass parameter one can specialized our result for the interesting de Sitter spacetime (de Sitter, 1917).

Even if an a priori numerical value for the ratio Δ (Eq. (81)) cannot be assigned, its appearance in various places like in Eqs. (86), (87), (89), (97), and (98) still allows the pseudoclassical theory to make quantitative predictions by comparing different physical processes in the regime where the semiclassical limit applies.

Supersymmetry and its local version—supergravity—are relevant in the fundamental theory of particle interactions. In modern particle theory, supersymmetry is the most general symmetry of the *S*-matrix consistent with relativistic quantum field theory (Haag *et al.*, 1975). So it is not inconceivable that nature might make some use of it. Indeed, superstrings (Green *et al.*, 1987; Schellekens, 1989) are the present best candidates for a consistent quantum theory unifying gravity with all other fundamental interactions, and supersymmetry appears to play a very important role for the quantum stability of superstring solutions in four-dimensional spacetime. In view of these reasons, the study of the geometry of the graded pseudomanifolds with both real number and anticommuting variables is well motivated.

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